

## ON COTORSION PAIRS OF CHAIN COMPLEXES

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ABSTRACT. In the paper we first construct a new cotorsion pair, in the category of chain complexes, from two given cotorsion pairs in the category of modules, and then we consider completeness of such pairs under certain conditions.

## 1. Introduction

Cotorsion pairs (or cotorsion theories) were invented by Salce in his study of abelian groups in [Sa]. However, the concept readily generalized to any abelian category, and its importance in homological algebra has been shown by its use in the proof of the flat cover conjecture [BBE]. The flat cover conjecture was positively settled by showing that the famous cotorsion pair  $(\mathcal{F}, \mathcal{C})$  is complete, where  $\mathcal{F}$  denotes the class of flat modules and  $\mathcal{C}$  denotes the class of cotorsion modules. On the other hand, there is a lot of interest in the complete cotorsion pairs in the category of chain complexes. It not only is used to show the existence of certain covers and envelopes in the category of chain complexes [AERO], but also is closely related to Quillen model structures and also to the existence of certain adjoints. In fact, a famous result of Hovey [Hov] says that a Quillen model structure on any abelian category  $\mathcal{C}$  is equivalent to two complete cotorsion pairs in  $\mathcal{C}$  which are compatible in a precise way. One of the upshots of this result was that the study of cotorsion pairs in the category of chain complexes attracted more attentions. Besides this, a recent result of a group of authors [BEJR, Theorem 3.5] shows that there is a tight connection between the complete cotorsion pairs in the category of chain complexes of modules and the existence of adjoint functors on the corresponding homotopy categories. Hence, there has been several attempts to get (complete) cotorsion pairs in  $\text{Ch}(R)$ , the category of chain complexes over a ring  $R$ , from ones in  $R\text{-Mod}$ , see e.g. [AERO], [AH], [BEJR], [EER], [GR], [G04], [G08], [YL].

Our main goal in this paper is first to construct a new cotorsion pair in  $\text{Ch}(R)$  from two given cotorsion pairs in  $R\text{-Mod}$ , and then to consider completeness of our constructed cotorsion pairs. More specifically, given two classes of  $R$ -modules  $\mathcal{U}$  and  $\mathcal{X}$ , where  $\mathcal{U} \subseteq \mathcal{X}$ , we have the following classes of chain complexes in  $\text{Ch}(R)$ .

- $dw\tilde{\mathcal{U}}$  is the class of all chain complexes  $U$  with each degree  $U_n \in \mathcal{U}$ .
- $ex\tilde{\mathcal{U}}$  is the class of all exact chain complexes  $U$  with each degree  $U_n \in \mathcal{U}$ .
- $\tilde{\mathcal{U}}$  is the class of all exact chain complexes  $U$  with each cycle  $Z_n U \in \mathcal{U}$ .

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**2010 Mathematics Subject Classification:** 16E05, 18G35.

**Key words:** chain complexes; cotorsion pairs.

This work was partly supported by NSF of China (Grant No. 11101197, 11201376, 11301240) and the Program of Science and Technique of Gansu Province (No. 145RJZA079).

- $\tilde{\mathcal{U}}_{\mathcal{X}}$  is the class of all exact chain complexes  $U$  with each degree  $U_n \in \mathcal{U}$  and each cycle  $Z_n U \in \mathcal{X}$ .

Then theorems 3.3 and 3.4 say that if  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  are two cotorsion pairs with  $\mathcal{U} \subseteq \mathcal{X}$  in  $R\text{-Mod}$ , then  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^{\perp})$  and  $({}^{\perp}(\tilde{\mathcal{Y}}_{\mathcal{V}}), \tilde{\mathcal{Y}}_{\mathcal{V}})$  are cotorsion pairs in  $\text{Ch}(R)$ . This result immediately yields a list of cotorsion pairs in  $\text{Ch}(R)$  below, and so our argument gives a unified proof for most of the existing cotorsion pairs in  $\text{Ch}(R)$ .

$$(\tilde{\mathcal{U}}, \tilde{\mathcal{U}}^{\perp}) \quad ({}^{\perp}\tilde{\mathcal{V}}, \tilde{\mathcal{V}}) \quad (ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^{\perp}) \quad ({}^{\perp}(ex\tilde{\mathcal{V}}), ex\tilde{\mathcal{V}})$$

Theorems 3.10 and 3.11 say the following: Assume that  $(\mathcal{U}, \mathcal{V})$  is a hereditary cotorsion pair in  $R\text{-Mod}$ . Then the cotorsion pair  $(dw\tilde{\mathcal{U}}, (dw\tilde{\mathcal{U}})^{\perp})$  is complete if and only if the cotorsion pair  $(ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^{\perp})$  is complete; and the cotorsion pair  $({}^{\perp}(dw\tilde{\mathcal{V}}), dw\tilde{\mathcal{V}})$  is complete if and only if the cotorsion pair  $({}^{\perp}(ex\tilde{\mathcal{V}}), ex\tilde{\mathcal{V}})$  is complete. In the end of this paper, we consider cogenerated sets of such pairs under certain conditions.

## 2. Preliminaries

Throughout this paper, let  $R$  be an associative ring with 1,  $R\text{-Mod}$  the category of left  $R$ -modules and  $\text{Ch}(R)$  the category of chain complexes of left  $R$ -modules.

We denote a chain complex  $\cdots \rightarrow C_{n+1} \xrightarrow{\delta_{n+1}^C} C_n \xrightarrow{\delta_n^C} C_{n-1} \rightarrow \cdots$  by  $(C, \delta)$  or simply  $C$ . The  $n$ th cycle of a chain complex  $C$  is defined as  $\text{Ker}(\delta_n^C)$  and is denoted by  $Z_n C$ , the  $n$ th boundary is  $\text{Im}(\delta_{n+1}^C)$  and is denoted by  $B_n C$ , the  $n$ th homology is the module  $H_n C = Z_n C / B_n C$ . A complex  $C$  is said to be exact if  $H_n C = 0$  for all  $n \in \mathbb{Z}$ .

We let  $S^n(M)$  denote the chain complex with all entries 0 except  $M$  in degree  $n$ , and let  $D^n(M)$  denote the chain complex  $C$  with  $C_n = C_{n-1} = M$  and all other entries 0, all differentials 0 except  $\delta_n = 1_M$ . The suspension of a chain complex  $C$ , denoted  $\Sigma C$ , is the chain complex given by  $(\Sigma C)_n = C_{n-1}$  and  $\delta_n^{\Sigma C} = -\delta_{n-1}^C$ . The chain complex  $\Sigma(\Sigma C)$  is denoted  $\Sigma^2 C$  and inductively we define  $\Sigma^n C$  for all  $n \in \mathbb{Z}$ .

Given two chain complexes  $X$  and  $Y$  we define  $\text{Hom}(X, Y)$  to be the complex of  $\mathbb{Z}$ -modules with  $n$ th degree  $\text{Hom}(X, Y)_n = \prod_{i \in \mathbb{Z}} \text{Hom}(X_i, Y_{i+n})$  and differential  $\delta_n$  satisfying  $(\delta_n(f))_i = \delta_{i+n}^Y f_i - (-1)^n f_{i-1} \delta_i^X$ . This gives a functor  $\text{Hom}(X, -) : \text{Ch}(R) \rightarrow \text{Ch}(\mathbb{Z})$  which is left exact, and exact if  $X_n$  is projective for all  $n$ . Similarly, the contravariant functor  $\text{Hom}(-, Y)$  sends right exact sequences to left exact sequences and is exact if  $Y_n$  is injective for all  $n$ . Note that the category  $\text{Ch}(R)$  is a Grothendieck category with a projective generator, and so it has enough projectives. Recall that a Grothendieck category is an abelian category with a generator and with the property that the direct limits are exact.

Recall that  $\text{Ext}_{\text{Ch}(R)}^1(X, Y)$  is the group of (equivalence classes) of short exact sequences  $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$  under the Baer sum. We let  $\text{Ext}_{dw}^1(X, Y)$  be the subgroup of  $\text{Ext}_{\text{Ch}(R)}^1(X, Y)$  consisting of those short exact sequences which are split in each degree. We often make use of the following standard fact.

**Lemma 2.1.** *For two chain complexes  $X$  and  $Y$ , we have*

$$\text{Ext}_{dw}^1(X, \Sigma^{-n-1}Y) \cong H_n \text{Hom}(X, Y) = \text{Ch}(R)(X, \Sigma^{-n}Y) / \sim,$$

where  $\sim$  is chain homotopy.

In particular, for two chain complexes  $X$  and  $Y$ ,  $\text{Hom}(X, Y)$  is exact if and only if for any  $n \in \mathbb{Z}$ , any  $f : \Sigma^n X \rightarrow Y$  is homotopic to 0 (or if and only if any  $f : X \rightarrow \Sigma^{-n} Y$  is homotopic to 0).

**Definition 2.2.** A pair  $(\mathcal{A}, \mathcal{B})$  in an abelian category  $\mathcal{C}$  is called a cotorsion pair if the following conditions hold:

- (1)  $\text{Ext}_{\mathcal{C}}^1(A, B) = 0$  for all  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ ;
- (2) If  $\text{Ext}_{\mathcal{C}}^1(A, X) = 0$  for all  $A \in \mathcal{A}$  then  $X \in \mathcal{B}$ ;
- (3) If  $\text{Ext}_{\mathcal{C}}^1(X, B) = 0$  for all  $B \in \mathcal{B}$  then  $X \in \mathcal{A}$ .

We think of a cotorsion pair  $(\mathcal{A}, \mathcal{B})$  as being “orthogonal with respect to  $\text{Ext}_{\mathcal{C}}^1$ ”. This is often expressed with the notation  $\mathcal{A} = {}^{\perp}\mathcal{B}$  and  $\mathcal{B} = \mathcal{A}^{\perp}$ . The notion of a cotorsion pair was first introduced by Salce in [Sa] and rediscovered by Enochs and coauthors in 1990’s. For a good reference on cotorsion pairs one can refer to [EJ].

**Definition 2.3.** A cotorsion pair  $(\mathcal{A}, \mathcal{B})$  in an abelian category  $\mathcal{C}$  is said to have enough projectives if for any object  $X \in \mathcal{C}$  there is a short exact sequence  $0 \rightarrow B \rightarrow A \rightarrow X \rightarrow 0$  with  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ . We say it has enough injectives if it satisfies the dual statement. If both of these hold we say the cotorsion pair is complete.

Note that if the category  $\mathcal{C}$  has enough injectives and projectives then a cotorsion pair  $(\mathcal{A}, \mathcal{B})$  is complete if and only if  $(\mathcal{A}, \mathcal{B})$  has enough injectives if and only if  $(\mathcal{A}, \mathcal{B})$  has enough projectives [EJ].

**Definition 2.4.** A cotorsion pair  $(\mathcal{A}, \mathcal{B})$  in an abelian category  $\mathcal{C}$  is said to be hereditary, if  $\text{Ext}_{\mathcal{C}}^i(A, B) = 0$  for any object  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$  and  $i \geq 1$ .

In  $R\text{-Mod}$ , the class of projectives is the left half of an obvious hereditary complete cotorsion pair while the class of injectives is the right half of an obvious hereditary complete cotorsion pair. There are many nontrivial examples of hereditary complete cotorsion pairs, which can be found in [GT]. We also need the next two definitions (see [St]).

**Definition 2.5.** Let  $\mathcal{S}$  be a class of objects of a Grothendieck category  $\mathcal{G}$ . An object  $X \in \mathcal{G}$  is called  $\mathcal{S}$ -filtered if there exists a well-ordered direct system  $(X_{\alpha}, i_{\alpha\beta} | \alpha < \beta \leq \sigma)$  indexed by an ordinal number  $\sigma$  such that

- (1)  $X_0 = 0$  and  $X_{\sigma} = X$ ,
- (2) for each limit ordinal  $\mu \leq \sigma$ , the direct limit of the subsystem  $(X_{\alpha}, i_{\alpha\beta} | \alpha < \beta \leq \mu)$  is precisely  $X_{\mu}$ , the direct limit morphisms being  $i_{\alpha\mu} : X_{\alpha} \rightarrow X_{\mu}$ ,
- (3)  $i_{\alpha\beta} : X_{\alpha} \rightarrow X_{\beta}$  is a monomorphism in  $\mathcal{G}$  for each  $\alpha < \beta \leq \sigma$ ,
- (4)  $\text{Coker}(i_{\alpha, \alpha+1}) \in \mathcal{S}$  for each  $\alpha < \sigma$ .

The direct system  $(X_{\alpha}, i_{\alpha\beta})$  is then called an  $\mathcal{S}$ -filtration of  $X$ . The class of all  $\mathcal{S}$ -filtered objects in  $\mathcal{G}$  is denoted by  $\text{Filt-}\mathcal{S}$ .

**Definition 2.6.** A class  $\mathcal{F}$  of objects in  $\mathcal{G}$  is called deconstructible if there is a set  $\mathcal{S}$  such that  $\mathcal{F} = \text{Filt-}\mathcal{S}$ .

If  $P$  is a projective  $R$ -module and  $x \in P$ , then Kaplansky [K] showed that there exists a countably generated summand of  $P$  which contains  $x$ . Enochs and López-Ramos generalized this ideal and introduced the notion of a Kaplansky class, see [EL, Definition 2.1].

**Definition 2.7.** A class  $\mathcal{K}$  of  $R$ -modules is called a  $\kappa$ -Kaplansky class if there exists a cardinal number  $\kappa$  such that for every  $M \in \mathcal{K}$  and for any subset  $S \subseteq M$  with  $\text{Card}(S) \leq \kappa$ , there exists a submodule  $N$  of  $M$  that contains  $S$  with the property that  $\text{Card}(N) \leq \kappa$  and both  $N$  and  $M/N$  are in  $\mathcal{K}$ . We say that  $\mathcal{K}$  is a Kaplansky class if it is a  $\kappa$ -Kaplansky class for some regular cardinal  $\kappa$ .

Let  $C$  be a chain complex in  $\text{Ch}(R)$ . By the cardinality of  $C$ ,  $\text{Card}(C)$ , we mean  $\text{Card}(\coprod_{n \in \mathbb{Z}} C_n)$ . By a subset  $S$  of  $X$  we mean a family  $(S_n)_{n \in \mathbb{Z}}$  such that  $S_n$  is a subset of  $C_n$ , for  $n \in \mathbb{Z}$ . Similarly, we have the notion of a Kaplansky class of chain complexes. We assume in the paper that all cardinals are regular, that is, are infinite cardinals which are not the sum of a smaller number of smaller cardinals. We let  $\omega$  denote the first limit ordinal.

#### ACKNOWLEDGEMENTS

The authors thank the referee for his/her careful reading and many considerable suggestions, which have improved the present paper.

### 3. Cotortion pairs in the category of chain complexes

**Lemma 3.1.** *Let  $X$  be a chain complex and  $J$  be an injective cogenerator for  $R\text{-Mod}$ . If every chain map  $\alpha : X \rightarrow S^n(J)$  lifts over  $D^n(J)$  for any  $n \in \mathbb{Z}$ , then  $X$  is exact.*

*Proof.* Let  $n$  be an arbitrary integer, we need only to show exactness of  $X$  in degree  $n$ . Suppose that  $t : X_n/B_nX \rightarrow J$  is a monomorphism. Then it is easy to check that  $\alpha : X \rightarrow S^n(J)$  is a chain map, where  $\alpha_i = 0$  for  $i \neq n$ , and  $\alpha_n$  is the composition of homomorphisms  $\pi_n : X_n \rightarrow X_n/B_nX$  and  $t$ . By hypothesis, there exists a chain map  $\beta : X \rightarrow D^n(J)$  such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ & \downarrow \alpha & \\ D^n(J) & \xrightarrow{p} & S^n(J) \longrightarrow 0 \end{array}$$

Put  $\delta_n^X = \rho_n \pi_n$ , where  $\rho_n : X_n/B_nX \rightarrow X_{n-1}$ . Thus  $t\pi_n = \alpha_n = p_n \beta_n = \beta_n = \beta_{n-1} \delta_n^X = \beta_{n-1} \rho_n \pi_n$ , and so  $t = \beta_{n-1} \rho_n$  since  $\pi_n$  is epic. This implies that  $\rho_n$  is a monomorphism. Therefore  $Z_nX = \text{Ker}(\rho_n \pi_n) = \text{Ker}(\pi_n) = B_nX$ . This proves exactness of  $X$ .  $\square$

**Definition 3.2.** Given two classes of  $R$ -modules  $\mathcal{U}$  and  $\mathcal{X}$  in  $R\text{-Mod}$  with  $\mathcal{U} \subseteq \mathcal{X}$ . We denote by  $\tilde{\mathcal{U}}_{\mathcal{X}}$  the class of all exact chain complexes  $U$  with each degree  $U_n \in \mathcal{U}$  and each cycle  $Z_nU \in \mathcal{X}$  in  $\text{Ch}(R)$ .

Clearly if we let  $\mathcal{U}$  and  $\mathcal{X}$  be certain classes of modules, we will get some familiar and interesting classes in  $\text{Ch}(R)$ . For example, if  $\mathcal{U} = \mathcal{P}$  is the class of all projective modules and  $\mathcal{X} = \mathcal{G}$  is the class of all Gorenstein projective modules in  $R\text{-Mod}$ , then  $\tilde{\mathcal{P}}_{\mathcal{G}}$  is the class of all complete projective resolutions of Gorenstein projective modules. (See [EJ] and [Hol] for Gorenstein projective modules).

**Theorem 3.3.** *Let  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  be two cotorsion pairs with  $\mathcal{U} \subseteq \mathcal{X}$  in  $R\text{-Mod}$ . Then  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^{\perp})$  is a cotorsion pair in  $\text{Ch}(R)$  and  $(\tilde{\mathcal{U}}_{\mathcal{X}})^{\perp}$  is the class of all chain complexes  $V$  for which each  $V_n \in \mathcal{V}$  and for which each map  $U \rightarrow V$  is null homotopic whenever  $U \in \tilde{\mathcal{U}}_{\mathcal{X}}$ .*

*Proof.* Let  $\widehat{\mathcal{W}}$  denote the class of all chain complexes  $V$  for which each  $V_n \in \mathcal{V}$  and for which each map  $U \rightarrow V$  is null homotopic whenever  $U \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ . It is clear that  $\widehat{\mathcal{W}}$  is closed under taking suspensions. Given any chain complex  $U \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ , and any  $R$ -module  $V \in \mathcal{V}$ , then by [G04, Lemma 3.1] we have  $\text{Ext}_{\text{Ch}(R)}^1(U, D^{n+1}(V)) \cong \text{Ext}_R^1(U_n, V) = 0$ , which implies that all disks  $D^n(V)$  are contained in  $\widehat{\mathcal{W}}$  whenever  $V \in \mathcal{V}$ . Similarly, for any  $U \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ , since  $U_n/B_n U \cong Z_{n-1}U \in \mathcal{X}$ , we get by [G08, Lemma 4.2] that  $\text{Ext}_{\text{Ch}(R)}^1(U, S^n(Y)) \cong \text{Ext}_R^1(U_n/B_n U, Y) = 0$  for any  $R$ -module  $Y \in \mathcal{Y}$ , and so each sphere  $S^n(Y) \in \widehat{\mathcal{W}}$  whenever  $Y \in \mathcal{Y}$ .

In the following we will show that  $(\widetilde{\mathcal{U}}_{\mathcal{X}}, \widehat{\mathcal{W}})$  is a cotorsion pair.

First suppose that  $U \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ , and  $W \in \widehat{\mathcal{W}}$ . Then any element  $0 \rightarrow W \rightarrow T \rightarrow U \rightarrow 0$  of  $\text{Ext}_{\text{Ch}(R)}^1(U, W)$  is degreewise split and so is an element of  $\text{Ext}_{dw}^1(U, W)$ . But it follows easily from Lemma 2.1 that  $\text{Ext}_{dw}^1(U, W) = 0$ . Thus  $\text{Ext}_{\text{Ch}(R)}^1(U, W) = 0$ .

Next assume that  $\text{Ext}_{\text{Ch}(R)}^1(U, C) = 0$  for all  $U \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ , we will show  $C \in \widehat{\mathcal{W}}$ . By [G04, Lemma 3.1], we have  $\text{Ext}_R^1(A, C_n) \cong \text{Ext}_{\text{Ch}(R)}^1(D^n(A), C) = 0$  since  $D^n(A)$  is clearly in  $\widetilde{\mathcal{U}}_{\mathcal{X}}$  whenever  $A$  is an  $R$ -module in  $\mathcal{U}$ . Thus  $C_n \in \mathcal{V}$ . Now let  $U \rightarrow C$  be a chain map, where  $U \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ . We would like to show that it is null homotopic. Clearly, we have  $\text{Ext}_{dw}^1(U, \Sigma^{-1}C) = \text{Ext}_{dw}^1(\Sigma U, C)$  and the last group equals 0 since  $\Sigma U \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ . Thus  $\text{Ext}_{dw}^1(U, \Sigma^{-1}C) = 0$ , and so  $C \in \widehat{\mathcal{W}}$  by Lemma 2.1.

Last we assume that  $\text{Ext}_{\text{Ch}(R)}^1(C, W) = 0$  for all  $W \in \widehat{\mathcal{W}}$ . We will show  $C \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ . Since for any  $R$ -module  $V \in \mathcal{V}$ , the disk  $D^{n+1}(V) \in \widehat{\mathcal{W}}$ , we have  $\text{Ext}_R^1(C_n, V) \cong \text{Ext}_{\text{Ch}(R)}^1(C, D^{n+1}(V)) = 0$ , and so  $C_n \in \mathcal{U}$ . Also since  $S^n(Y) \in \widehat{\mathcal{W}}$  for any  $R$ -module  $Y \in \mathcal{Y}$ , we have  $\text{Ext}_{\text{Ch}(R)}^1(C, S^n(Y)) = 0$ , and so  $\text{Ext}_R^1(C_n/B_n C, Y) = 0$  by [G08, Lemma 4.2], which implies that each  $C_n/B_n C$  belongs to  $\mathcal{X}$ . But by using Lemma 3.1 we get that each  $Z_n C \cong C_{n+1}/B_{n+1} C \in \mathcal{X}$ . Thus  $C \in \widetilde{\mathcal{U}}_{\mathcal{X}}$ , as desired.  $\square$

Given two cotorsion pairs  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  in  $R\text{-Mod}$ . Clearly,  $\mathcal{U} \subseteq \mathcal{X}$  if and only if  $\mathcal{Y} \subseteq \mathcal{V}$ , with this in mind, we also have the following result.

**Theorem 3.4.** *Let  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  be two cotorsion pairs with  $\mathcal{U} \subseteq \mathcal{X}$  in  $R\text{-Mod}$ . Then  $({}^\perp(\widetilde{\mathcal{Y}}_{\mathcal{V}}), \widetilde{\mathcal{Y}}_{\mathcal{V}})$  is a cotorsion pair in  $\text{Ch}(R)$  and  ${}^\perp(\widetilde{\mathcal{Y}}_{\mathcal{V}})$  is the class of all chain complexes  $X$  for which each  $X_n \in \mathcal{X}$  and for which each map  $X \rightarrow Y$  is null homotopic whenever  $Y \in \widetilde{\mathcal{Y}}_{\mathcal{V}}$ .*

*Proof.* It is dual to the proof of Theorem 3.3.  $\square$

**Definition 3.5.** Given a class of  $R$ -modules  $\mathcal{A}$ . We define the following classes of chain complexes in  $\text{Ch}(R)$ .

- (1)  $dw\widetilde{\mathcal{A}}$  is the class of all chain complexes  $A$  with each degree  $A_n \in \mathcal{A}$ .
- (2)  $ex\widetilde{\mathcal{A}}$  is the class of all exact chain complexes  $A$  with each degree  $A_n \in \mathcal{A}$ .
- (3)  $\widetilde{\mathcal{A}}$  is the class of all exact chain complexes  $A$  with each cycle  $Z_n A \in \mathcal{A}$ .

The “ $dw$ ” is meant to stand for “degreewise” while the “ $ex$ ” is meant to stand for “exact”.

Moreover, if we are given any cotorsion pair  $(\mathcal{U}, \mathcal{V})$  in  $R\text{-Mod}$ , then following [G04] we will denote  $\widetilde{\mathcal{U}}^\perp$  by  $dg\widetilde{\mathcal{V}}$  and  ${}^\perp\widetilde{\mathcal{V}}$  by  $dg\widetilde{\mathcal{U}}$ .

The next two corollaries are contained in [G04, Proposition 3.6], and [G08, Proposition 3.3], respectively, but the author considered them on a general abelian category. Here we present short proofs of them for our case.

**Corollary 3.6.** *Let  $(\mathcal{U}, \mathcal{V})$  be a cotorsion pair in  $R\text{-Mod}$ . Then  $(\tilde{\mathcal{U}}, dg\tilde{\mathcal{V}})$  and  $(dg\tilde{\mathcal{U}}, \tilde{\mathcal{V}})$  are cotorsion pairs in  $Ch(R)$ .*

*Proof.* We just prove one of the statements since the other is dual. Note that  $(\mathcal{X}, \mathcal{Y}) = (\mathcal{U}, \mathcal{V})$  is another cotorsion pair with  $\mathcal{U} \subseteq \mathcal{X}$ . So, by Theorem 3.3,  $(\tilde{\mathcal{U}}, \tilde{\mathcal{U}}^\perp)$  is a cotorsion pair since clearly  $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_{\mathcal{U}}$ .  $\square$

In the following, we take  $(\mathcal{P}, \mathcal{M})$  and  $(\mathcal{M}, \mathcal{I})$  as the usual projective and injective cotorsion pairs in  $R\text{-Mod}$ , where  $\mathcal{P}$  denotes the class of all projective  $R$ -modules,  $\mathcal{M}$  denotes the class of all  $R$ -modules, and  $\mathcal{I}$  denotes the class of all injective  $R$ -modules. Note that for any cotorsion pair  $(\mathcal{U}, \mathcal{V})$  in  $R\text{-Mod}$  we always have inclusions  $\mathcal{P} \subseteq \mathcal{U} \subseteq \mathcal{M}$  and  $\mathcal{I} \subseteq \mathcal{V} \subseteq \mathcal{M}$ .

**Corollary 3.7.** *Let  $(\mathcal{U}, \mathcal{V})$  be a cotorsion pair in  $R\text{-Mod}$ . Then  $(ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^\perp)$  and  $(^\perp(ex\tilde{\mathcal{V}}), ex\tilde{\mathcal{V}})$  are cotorsion pairs in  $Ch(R)$ .*

*Proof.* Again we will just prove one of the statements since the other is dual. Note that the injective cotorsion pair  $(\mathcal{M}, \mathcal{I})$  in  $R\text{-Mod}$  is another one such that  $\mathcal{U} \subseteq \mathcal{M}$ . So, by Theorem 3.3,  $(ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^\perp)$  is a cotorsion pair since clearly  $ex\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_{\mathcal{M}}$ .  $\square$

**Remark 3.8.** According to [YL], the induced cotorsion pairs  $(\tilde{\mathcal{U}}, dg\tilde{\mathcal{V}})$  and  $(dg\tilde{\mathcal{U}}, \tilde{\mathcal{V}})$  are both complete when the given cotorsion pair  $(\mathcal{U}, \mathcal{V})$  is hereditary and complete. In particular,  $(dg\tilde{\mathcal{P}}, \tilde{\mathcal{M}})$  and  $(\tilde{\mathcal{M}}, dg\tilde{\mathcal{I}})$  are complete cotorsion pairs in  $Ch(R)$ , where  $\tilde{\mathcal{M}}$  denotes the class of all exact chain complexes.

**Remark 3.9.** According to [G08], if we have a cotorsion pair  $(\mathcal{U}, \mathcal{V})$  in  $R\text{-Mod}$ , then  $(dw\tilde{\mathcal{U}}, (dw\tilde{\mathcal{U}})^\perp)$  and  $(^\perp(dw\tilde{\mathcal{V}}), dw\tilde{\mathcal{V}})$  are cotorsion pairs in  $Ch(R)$ .

The following result shows that there are intimate connections of completeness between the induced cotorsion pairs  $(dw\tilde{\mathcal{U}}, (dw\tilde{\mathcal{U}})^\perp)$  and  $(ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^\perp)$ .

**Theorem 3.10.** *Assume that  $(\mathcal{U}, \mathcal{V})$  is a hereditary cotorsion pair in  $R\text{-Mod}$ . Then  $(dw\tilde{\mathcal{U}}, (dw\tilde{\mathcal{U}})^\perp)$  is complete if and only if  $(ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^\perp)$  is complete.*

*Proof.*  $(\Rightarrow)$ . Since the cotorsion pair  $(dw\tilde{\mathcal{U}}, (dw\tilde{\mathcal{U}})^\perp)$  is complete, for any chain complex  $C$ , there exists an exact sequence  $0 \rightarrow V \rightarrow U \rightarrow C \rightarrow 0$  such that  $U \in dw\tilde{\mathcal{U}}$  and  $V \in (dw\tilde{\mathcal{U}})^\perp$ . By Remark 3.8, there is an exact sequence  $0 \rightarrow I \rightarrow E \rightarrow U \rightarrow 0$  with  $E$  exact and  $I \in dg\tilde{\mathcal{I}}$ . Now we consider the pull-back diagram as

follows:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & I & \xlongequal{\quad} & I & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & V & \longrightarrow & U & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Since  $I \in dg\tilde{\mathcal{L}}$ , we have  $I \in (ex\tilde{\mathcal{U}})^\perp$ . Clearly, we have  $V \in (ex\tilde{\mathcal{U}})^\perp$  since  $V \in (dw\tilde{\mathcal{U}})^\perp$ , and so the exactness of the leftmost column of the above diagram implies  $X \in (ex\tilde{\mathcal{U}})^\perp$ . By hypothesis again, there is an exact sequence  $0 \rightarrow V' \rightarrow U' \rightarrow E \rightarrow 0$  such that  $U' \in dw\tilde{\mathcal{U}}$  and  $V' \in (dw\tilde{\mathcal{U}})^\perp$ . Again consider the following pull-back diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & V' & \xlongequal{\quad} & V' & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & Y & \longrightarrow & U' & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Since  $E$  is exact and  $V' \in (dw\tilde{\mathcal{U}})^\perp$  is easily seen exact, we get that  $U'$  is exact and so  $U' \in ex\tilde{\mathcal{U}}$ . Furthermore, since  $V' \in (dw\tilde{\mathcal{U}})^\perp \subseteq (ex\tilde{\mathcal{U}})^\perp$  and  $X \in (ex\tilde{\mathcal{U}})^\perp$ , the exactness of the leftmost column of the above diagram implies  $Y \in (ex\tilde{\mathcal{U}})^\perp$ . Now the second exact row of the above diagram implies that the cotorsion pair  $(ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^\perp)$  has enough projectives, and so it is complete.

( $\Leftarrow$ ). Let  $C$  be any chain complex. Then there is an exact sequence  $0 \rightarrow H \rightarrow G \rightarrow C \rightarrow 0$  with  $G \in ex\tilde{\mathcal{U}}$  and  $H \in (ex\tilde{\mathcal{U}})^\perp$  since the cotorsion pair  $(ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^\perp)$  is complete. By Remark 3.8, there is an exact sequence  $0 \rightarrow H \rightarrow E \rightarrow P \rightarrow 0$

with  $E$  exact and  $P \in dg\tilde{\mathcal{P}}$ . Consider the following push-out diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & E & \longrightarrow & D & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & P & \xlongequal{\quad} & P & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

Since  $G \in ex\tilde{\mathcal{U}} \subseteq dw\tilde{\mathcal{U}}$  and  $P$  is easily seen in  $dw\tilde{\mathcal{U}}$ , we get that  $D \in dw\tilde{\mathcal{U}}$ . Since the cotorsion pair  $(ex\tilde{\mathcal{U}}, (ex\tilde{\mathcal{U}})^\perp)$  is complete, there is an exact sequence  $0 \rightarrow E \rightarrow H' \rightarrow G' \rightarrow 0$  with  $H' \in (ex\tilde{\mathcal{U}})^\perp$  and  $G' \in ex\tilde{\mathcal{U}}$ . Now consider the following push-out diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & D & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & H' & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & G' & \xlongequal{\quad} & G' & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

Since  $E$  and  $G'$  are exact, so is  $H'$ . Thus it follows from [EJ, Lemma 7.4.1] that  $H' \in (ex\tilde{\mathcal{U}})^\perp \cap \tilde{\mathcal{M}} = (dw\tilde{\mathcal{U}})^\perp$ . It is not hard to see that  $B \in dw\tilde{\mathcal{U}}$ . This proves that the cotorsion pair  $(dw\tilde{\mathcal{U}}, (dw\tilde{\mathcal{U}})^\perp)$  is complete.  $\square$

Dually, we have the following result without giving its proof.

**Theorem 3.11.** *Assume that  $(\mathcal{U}, \mathcal{V})$  is a hereditary cotorsion pair in  $R\text{-Mod}$ . Considering the statements below. Then  $({}^\perp(dw\tilde{\mathcal{V}}), dw\tilde{\mathcal{V}})$  is complete if and only if  $({}^\perp(ex\tilde{\mathcal{V}}), ex\tilde{\mathcal{V}})$  is complete.*

Assume that the given cotorsion pairs  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  with  $\mathcal{U} \subseteq \mathcal{X}$  in  $R\text{-Mod}$  are hereditary, then it is easily seen that the induced cotorsion pairs  $(\tilde{\mathcal{U}}_\mathcal{X}, (\tilde{\mathcal{U}}_\mathcal{X})^\perp)$  and  $({}^\perp(\tilde{\mathcal{Y}}_\mathcal{V}), \tilde{\mathcal{Y}}_\mathcal{V})$  are both hereditary. In the following, we are ready to show that our induced cotorsion pairs are complete under certain conditions. We will use a generalized version, of a well-known result of Eklof and Trlifaj [ET, Theorem



10], which says that every cotorsion pair  $(\mathcal{A}, \mathcal{B})$  in any Grothendieck category with enough projectives is complete if it is cogenerated by a set, see [Hov, Section 6]. We say a cotorsion pair  $(\mathcal{A}, \mathcal{B})$  in an abelian category  $\mathcal{C}$  is cogenerated by a set if there is a set  $\mathcal{S} \subseteq \mathcal{A}$  such that  $\mathcal{S}^\perp = \mathcal{B}$ .

**Proposition 3.12.** *Let  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  be two cotorsion pairs with  $\mathcal{U} \subseteq \mathcal{X}$  in  $R\text{-Mod}$ . If  $(\mathcal{U}, \mathcal{V})$  is cogenerated by a set  $\{A_i | i \in I\}$ , and  $(\mathcal{X}, \mathcal{Y})$  is cogenerated by a set  $\{B_j | j \in J\}$ , then the induced cotorsion pair  $({}^\perp(\tilde{\mathcal{Y}}_{\mathcal{V}}), \tilde{\mathcal{Y}}_{\mathcal{V}})$  is cogenerated by the set  $\mathcal{S} = \{S^n(R) | n \in \mathbb{Z}\} \cup \{S^n(A_i) | n \in \mathbb{Z}, i \in I\} \cup \{D^n(B_j) | n \in \mathbb{Z}, j \in J\}$ , and so it is complete.*

*Proof.* Dual to the proof of Theorem 3.3, we can prove that each sphere  $S^n(U) \in {}^\perp(\tilde{\mathcal{Y}}_{\mathcal{V}})$  whenever  $U \in \mathcal{U}$ , and each disk  $D^n(X) \in {}^\perp(\tilde{\mathcal{Y}}_{\mathcal{V}})$  whenever  $X \in \mathcal{X}$ . Thus we have  $\mathcal{S} \subseteq {}^\perp(\tilde{\mathcal{Y}}_{\mathcal{V}})$ , and so  $\mathcal{S}^\perp \supseteq ({}^\perp(\tilde{\mathcal{Y}}_{\mathcal{V}}))^\perp = \tilde{\mathcal{Y}}_{\mathcal{V}}$ . To see the reverse inclusion, now suppose  $Y \in \mathcal{S}^\perp$ . Then  $\text{Ext}_{\text{Ch}(R)}^1(S^n(A_i), Y) = 0$  for all  $i \in I$ . Since  $\text{Ext}_{\text{Ch}(R)}^1(S^n(A_i), Y) \cong \text{Ext}_R^1(A_i, Z_n Y)$  by [G04, Lemma 3.1], and the cotorsion pair  $(\mathcal{U}, \mathcal{V})$  is cogenerated by  $\{A_i | i \in I\}$ , we get that each  $Z_n Y \in \mathcal{V}$ .

Next we show that  $Y$  is exact. If we apply  $\text{Hom}_{\text{Ch}(R)}(-, Y)$  to the short exact sequence  $0 \rightarrow S^{n-1}(R) \rightarrow D^n(R) \rightarrow S^n(R) \rightarrow 0$ , then we have an induced exact sequence of abelian groups

$$\text{Hom}_{\text{Ch}(R)}(D^n(R), Y) \rightarrow \text{Hom}_{\text{Ch}(R)}(S^{n-1}(R), Y) \rightarrow \text{Ext}_{\text{Ch}(R)}^1(S^n(R), Y) = 0.$$

This means that every chain map  $S^{n-1}(R) \rightarrow Y$  can be extended to  $D^n(R)$ . So  $Y$  is exact by [G08, Lemma 2.4].

It is left to show that each degree  $Y_n$  of  $Y$  belongs to  $\mathcal{Y}$  for any integer  $n \in \mathbb{Z}$ . By [G04, Lemma 3.1], we have  $\text{Ext}_R^1(B_j, Y_n) \cong \text{Ext}_{\text{Ch}(R)}^1(D^n(B_j), Y) = 0$  for all  $j \in J$ . Thus  $Y_n \in \mathcal{Y}$  since the cotorsion pair  $(\mathcal{X}, \mathcal{Y})$  is cogenerated by  $\{B_j | j \in J\}$ . This shows that  $\mathcal{S}$  cogenerates the cotorsion pair  $({}^\perp(\tilde{\mathcal{Y}}_{\mathcal{V}}), \tilde{\mathcal{Y}}_{\mathcal{V}})$ .  $\square$

**Proposition 3.13.** *Let  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  be two cotorsion pairs with  $\mathcal{U} \subseteq \mathcal{X}$  in  $R\text{-Mod}$ . If both  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  are cogenerated by sets, then so is the induced cotorsion pair  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^\perp)$ , and so it is complete.*

*Proof.* Since  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  are cogenerated by sets, the two classes  $\mathcal{U}$  and  $\mathcal{X}$  are deconstructible by [St]. Then  $dw\tilde{\mathcal{U}}$  and  $\tilde{\mathcal{X}}$  are deconstructible classes by [St, Theorem 4.2], and so  $\tilde{\mathcal{U}}_{\mathcal{X}} = dw\tilde{\mathcal{U}} \cap \tilde{\mathcal{X}}$  is deconstructible by [St, Proposition 2.9(2)]. This implies that the cotorsion pair  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^\perp)$  is cogenerated by a set, and so it is complete.  $\square$

It is known that every Kaplansky class which is closed under well ordered direct limits is deconstructible, and every deconstructible class is Kaplansky. However, both of the converse do not hold in general (see [HT, Lemmas 6.7 and 6.9, and Example 6.8]). In the following we present two examples relating to Kaplansky classes as applications of Proposition 3.13.

**Example 3.14.** Let  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  be two cotorsion pairs with  $\mathcal{U} \subseteq \mathcal{X}$  in  $R\text{-Mod}$ . If  $\mathcal{U}$  and  $\mathcal{X}$  are Kaplansky classes which are both closed under well ordered direct limits, and  $\mathcal{X}$  is resolving, then  $\tilde{\mathcal{U}}_{\mathcal{X}}$  is a Kaplansky class of chain complexes which is closed under well ordered direct limits. Thus, the cotorsion pair  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^\perp)$  is cogenerated by a set, and so it is complete.

*Proof.* Let  $\kappa_1, \kappa_2$  be the cardinal numbers such that  $\mathcal{U}$  is  $\kappa_1$ -Kaplansky, and  $\mathcal{X}$  is  $\kappa_2$ -Kaplansky. Let  $\kappa$  be a cardinal number larger than  $\max\{\kappa_1, \kappa_2, \omega, \text{Card}(R)\}$ . In the following, we wish to show that  $\tilde{\mathcal{U}}_{\mathcal{X}}$  is  $\kappa$ -Kaplansky. So assume that  $U \in \tilde{\mathcal{U}}_{\mathcal{X}}$  and  $S$  is a subset of  $U$  with  $\text{Card}(S) \leq \kappa$ . We show that there exists a chain subcomplex  $W$  of  $U$  such that  $S \subseteq W$ ,  $\text{Card}(W) \leq \kappa$ , and  $W$  and  $U/W$  are contained in  $\tilde{\mathcal{U}}_{\mathcal{X}}$ .

Note that  $\tilde{\mathcal{U}}_{\mathcal{M}}$  and  $\tilde{\mathcal{X}}_{\mathcal{X}}$  are two  $\kappa$ -Kaplansky classes of chain complexes by [AH, Theorem 3.4]. Thus there exists a chain subcomplex  $U^1$  of  $U$  such that  $S \subseteq U^1$ ,  $\text{Card}(U^1) \leq \kappa$ , and  $U^1$  and  $U/U^1$  are contained in  $\tilde{\mathcal{U}}_{\mathcal{M}}$ . Again, since  $\tilde{\mathcal{X}}_{\mathcal{X}}$  is  $\kappa$ -Kaplansky, there exists a chain subcomplex  $U^2$  of  $U$  such that  $U^1 \subseteq U^2$ ,  $\text{Card}(U^2) \leq \kappa$ , and  $U^2$  and  $U/U^2$  are contained in  $\tilde{\mathcal{X}}_{\mathcal{X}}$ . Thus we will construct inductively  $\{U^i\}_{i \in \mathbb{N}}$  of chain subcomplexes of  $U$ , satisfying the following three properties:

- For any two integers  $i, j \in \mathbb{N}$  with  $i < j$ ,  $U^i$  is a chain subcomplex of  $U^j$ ;
- $U^i$  satisfies  $\text{Card}(U^i) \leq \kappa$ , and  $U^i, U/U^i \in \tilde{\mathcal{U}}_{\mathcal{M}}$  whenever  $i \in \mathbb{N}$  is odd;
- $U^i$  satisfies  $\text{Card}(U^i) \leq \kappa$ , and  $U^i, U/U^i \in \tilde{\mathcal{X}}_{\mathcal{X}}$  whenever  $i \in \mathbb{N}$  is even.

If we take  $W = \varinjlim_{i \in \mathbb{N}} U^i = \bigcup_{i \in \mathbb{N}} U^i$ , then we see that the complex  $W$  is exact because of exactness of each  $U^i$ . Clearly,  $W_n = \varinjlim_{i \in \mathbb{N}} U_n^i = \bigcup_{i \in \mathbb{N}} U_n^i = \bigcup_{i \in \mathbb{N}} U_n^{2i-1}$ . But by constructions, we have each  $U^{2i-1} \in \tilde{\mathcal{U}}_{\mathcal{M}}$ . In particular,  $W_n \in \mathcal{U}$ . Furthermore, each  $Z_n W \in \mathcal{X}$  since  $Z_n W = \varinjlim_{i \in \mathbb{N}} Z_n U^i = \bigcup_{i \in \mathbb{N}} Z_n U^i = \bigcup_{i \in \mathbb{N}} Z_n U^{2i}$  and  $U^{2i} \in \tilde{\mathcal{X}}_{\mathcal{X}}$  by constructions. Therefore the chain subcomplex  $W$  of  $U$  satisfies  $W \in \tilde{\mathcal{U}}_{\mathcal{X}}$ ,  $S \subseteq W$ , and of course  $\text{Card}(W) \leq \kappa$ . To finish the proof, we need only argue that  $U/W \in \tilde{\mathcal{U}}_{\mathcal{X}}$ . It follows from the short exact sequence  $0 \rightarrow W \rightarrow U \rightarrow U/W \rightarrow 0$  of chain complexes that  $U/W$  is exact. Also one can check easily that  $U/W = U/(\varinjlim_{i \in \mathbb{N}} U^i) \cong \varinjlim_{i \in \mathbb{N}} U/U^i$ , and then an easy computation shows that  $U_n/W_n \in \mathcal{U}$  and  $Z_n(U/W) \in \mathcal{X}$ . This shows that  $\tilde{\mathcal{U}}_{\mathcal{X}}$  is a Kaplansky class. Note that  $\tilde{\mathcal{U}}_{\mathcal{X}}$  is also closed under well ordered direct limits, and so it is deconstructible. This implies that the cotorsion pair  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^{\perp})$  is cogenerated by a set.  $\square$

Recall that an exact sequence  $0 \rightarrow L \rightarrow M$  of  $R$ -modules is pure if  $0 \rightarrow N \otimes_R L \rightarrow N \otimes_R M$  is exact for any right  $R$ -module  $N$ . We say a submodule  $L$  of  $M$  is pure if the sequence  $0 \rightarrow L \rightarrow M$  is pure exact. Similarly, we have the notion of a pure chain subcomplexes, but this will use tensor product of chain complexes (see [GR]). It is easy to see that if a class of  $R$ -modules is closed under pure submodules and cokernels of pure monomorphisms, then it is Kaplansky, also it is deconstructible.

**Example 3.15.** Let  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{X}, \mathcal{Y})$  be two cotorsion pairs with  $\mathcal{U} \subseteq \mathcal{X}$  in  $R\text{-Mod}$ . Assume that  $\mathcal{U}$  and  $\mathcal{X}$  are both closed under pure submodules and cokernels of pure monomorphisms. Then  $\tilde{\mathcal{U}}_{\mathcal{X}}$  is closed under pure subcomplexes and cokernels of pure monomorphisms, and so the cotorsion pair  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^{\perp})$  is cogenerated by a set. Thus  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^{\perp})$  is complete..

*Proof.* Suppose that the exact sequence  $0 \rightarrow V \rightarrow U \rightarrow U/V \rightarrow 0$  is pure in  $\text{Ch}(R)$  with  $U \in \tilde{\mathcal{U}}_{\mathcal{X}}$ . Then chain complexes  $V$  and  $U/V$  are exact as shown in proof of [WL, Lemma 2.7], and so for each  $n \in \mathbb{Z}$ ,  $0 \rightarrow V_n \rightarrow U_n \rightarrow U_n/V_n \rightarrow 0$  and  $0 \rightarrow Z_n V \rightarrow Z_n U \rightarrow Z_n(U/V) \rightarrow 0$  are pure exact in  $R\text{-Mod}$  by [WL, Lemmas 2.6 and 3.7]. It is easily seen that  $V$  and  $U/V$  are in  $\tilde{\mathcal{U}}_{\mathcal{X}}$ . This shows that  $\tilde{\mathcal{U}}_{\mathcal{X}}$

is closed under pure subcomplexes and cokernels of pure monomorphisms. Now it follows that the cotorsion pair  $(\tilde{\mathcal{U}}_{\mathcal{X}}, (\tilde{\mathcal{U}}_{\mathcal{X}})^{\perp})$  is cogenerated by a set since  $\tilde{\mathcal{U}}_{\mathcal{X}}$  is clearly deconstructible.  $\square$

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